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The Solution of a Problem in the Theory of Epidemics

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In a 1971 paper, Hoppensteadt and Waltman consider a deterministic epidemic model that accounts for certain threshold phenomena occurring in the spread of infection. A system of nonlinear delay integral equations describe this model. We describe a method for constructing functions which approximate the solution of the system of integral equations. The approximating functions are shown to exist and to converge to the solution of the system.

1. INTRODUCTION

A deterministic model of epidemics was proposed by Cooke [3] and studied by Hoppensteadt and Waltman [5]. In [6], Hoppensteadt and Waltman made a major modification to the model presented in [5] and studied the resulting generalization. This work considers the model presented in [6]. The principal feature of this model is that it accounts for certain threshold phenomena involved in the spread of infection. The model also allows the study of epidemics in which susceptible individuals become infected, recover from the infection, enjoy a certain period of immunity, and then return to the susceptible state.

The model gives rise to a system of integral equations which involve a time lag. Since analytical methods for solving this system are, in general, not available, numerical techniques are desirable. This work concerns a method for approximating the solution to the system of integral equations.

We now give a brief description of the model studied in [6].

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At time t, the population is divided into four disjoint classes. These are:

$S(t)$ the number of individuals susceptible to the infection but not yet exposed,

$E(t)$ the number of individuals exposed to the infection but not yet infectious,

$I(t)$ the number of infective individuals,

$R(t)$ the number of individuals who have recovered from the infection and are immune.

It is assumed the infection spreads according to the following rules.

(i) The number of susceptibles exposed to the infection in a time interval t to $t + h$ is

$$\int_t^{t+h} r(x)\, I(x)\, S(x)\, dx,$$

where r is a positive, continuous proportionality function. That is, the rate of exposure of susceptibles is proportional to the product of the number of infectives and the number of susceptibles.

(ii) An individual who is first exposed at time τ becomes infective at time t if

$$\int_\tau^t [\rho_1(x) + \rho_2(x)\, I(x)]\, dx = m,$$

where ρ_1 and ρ_2 are given continuous nonnegative functions and m is a nonnegative constant.

(iii) An individual infected at time t becomes immune at time $t + \sigma$, σ a positive constant.

(iv) An individual who becomes immune at time t becomes susceptible at time $t + \omega$, ω a nonnegative constant.

(v) The population size $P = S(t) + E(t) + I(t) + R(t)$ is constant.

In addition to these rules certain initial conditions are required. It is assumed that at some time, say $t = 0$, a number of infectious individuals are introduced into an initial susceptible population of S_0 individuals. The history of the infectious individuals is assumed to be described by a given function I_0 defined on the interval $-\sigma \leqslant t \leqslant 0$. Since the infection lasts a fixed time σ in an individual, no member of the initial infectives became infective before time $t = -\sigma$; therefore it is assumed that $I_0(-\sigma) = 0$. No member may recover during the interval $[-\sigma, 0]$ so I_0 is nondecreasing on this interval. Assuming (iii) applies to these initial infectives, the future of this group is known. Taking this into account an extension of I_0 to the real line is defined by

$$\begin{aligned} I_0(t) &= 0, & |t| &> \sigma \\ &= I_0(t), & -\sigma &\leqslant t \leqslant 0 \\ &= I_0(0) - I_0(t-\sigma), & 0 &\leqslant t \leqslant \sigma. \end{aligned}$$

If the infection is to spread through the population, there must exist a finite number t_0 such that

$$\int_0^{t_0} [\rho_1(x) + \rho_2(x)\, I_0(x)]\, dx = m.$$

This is the threshold condition. A nondecreasing continuous function I_0, defined on $[-\sigma, 0]$ such that $I_0(-\sigma) = 0$, whose extension satisfies the above integral condition is called an *admissible function*.

In [6] it is shown that under the assumptions above, the epidemic is described by a system of five integral equations of which equations (1.1)–(1.3) are sufficient to determine the behavior of the system (the remaining variables being described in terms of these).

$$\begin{aligned} \int_{\tau(t)}^{t} [\rho_1(x) + \rho_2(x)\, I(x)]\, dx &= m, & t &\geqslant t_0 \\ \tau(t) &= 0, & t &\leqslant t_0, \end{aligned} \tag{1.1}$$

$$S(t) = I_1(t) + S_0 - \int_{\tau(t-\sigma-\omega)}^{t} r(x)\, I(x)\, S(x)\, dx, \tag{1.2}$$

$$I(t) = I_0(t) + \int_{\tau(t-\sigma)}^{\tau(t)} r(x)\, I(x)\, S(x)\, dx, \tag{1.3}$$

$$E(t) = \int_{\tau(t)}^{t} r(x)\, I(x)\, S(x)\, dx, \tag{1.4}$$

and

$$R(t) = I_2(t) + \int_{\tau(t-\sigma-\omega)}^{\tau(t-\sigma)} r(x)\, I(x)\, S(x)\, dx, \tag{1.5}$$

where

$$\begin{aligned} I_1(t) &= 0, & t &\leqslant \omega, \\ &= I_0(0) - I_0(t-\omega), & t &\geqslant \omega \end{aligned}$$

and

$$\begin{aligned} I_2(t) &= I_0(0) - I_0(t), & t &\leqslant \omega, \\ &= I_0(t-\omega) - I_0(t), & t &\geqslant \omega \end{aligned}$$

Hoppensteadt and Waltman studied equations (1.1)–(1.3) under separate conditions on the coefficient functions ρ_1 and ρ_2. The conditions are

H1: The function ρ_2 is positive for all $t \geqslant 0$.

H2: The function ρ_1 is positive for all $t \geqslant 0$.

Their main result is the following:

Let I_0 be an admissible function and let r, ρ_1 and ρ_2 be continuous functions with r positive. Suppose either condition H1 holds with $I_0(t_0) > 0$ or condition H2 holds. Then there exist unique continuous, nonnegative functions τ, I, S, E and R which satisfy (1.1)–(1.5) for all nonnegative t. Moreover, this solution depends continuously on the choice of r, ρ_1, ρ_2, m and the initial conditions S_0 and I_0.

We assume throughout this work that the hypothesis of the preceding theorem is satisfied. The organization of the remainder of this paper is as follows. The method of approximating the solution of (1.1)–(1.3) is described in Section 2. Results concerning existence and convergence of the approximating functions are stated and proved in Section 3. This work complements that given in [2]. One point which should be emphasized here is that the approximating functions which we construct are physically realistic. In particular, the approximations to I and S are nonnegative and bounded above by the population size.

2. A Polygonal Model

Solving the system of equations (1.1)–(1.3) on a closed finite interval $[0, c]$, $c > t_0$, is a difficult task. However, a related problem which is amenable to solution may be formed by using the technique suggested by Haymond in [4]. This technique consists of modeling the functions τ, I and S by polygons which are required to satisfy (1.1)–(1.3) at their knots. It is, of course, hoped that the resulting polygons will be an adequate approximation to the solution of (1.1)–(1.3). In this section we describe the modeling of τ, I and S by polygons and in the next we state conditions under which such polygons exist.

First, suppose I_0 is an admissible function and that t_0 has been determined by solving the equation

$$\int_0^{t_0} [\rho_1(x) + \rho_2(x) I_0(x)]\, dx = m. \tag{2.1}$$

Since I_0 is an admissible function, a solution t_0 of this equation exists. Let W denote partition

$$0 = \xi_0 < \xi_1 < \cdots < \xi_L = t_0 < \xi_{L+1} < \cdots < \xi_N = c, \qquad (N \geqslant 3),$$

of the interval $[0, c]$ and let $Y(W)$ denote the space of all real polygons which have knots at the points of W. For t in $[\xi_k, \xi_{k+1}]$, y in $Y(W)$ has the representation

$$y(t) = y_k(t) = C_{k,0} \frac{t - \xi_{k+1}}{\xi_k - \xi_{k+1}} + C_{k,1} \frac{t - \xi_k}{\xi_{k+1} - \xi_k},$$

$k = 0, 1,..., N - 1$, for some constants $C_{k,0}$, $C_{k,1}$. Since y is continuous on $[0, c]$,

$$C_{k,0} = y_{k-1}(\xi_k), \qquad k = 0, 1,..., N - 1,$$

where $y_{-1}(\xi_0)$ is given by the initial conditions.

For $0 \leqslant t \leqslant t_0$, we need only model S by a polygon since on this interval $\tau(t) = 0$ and $I(t) = I_0(t)$ are known. We will make it a practice to denote by f_h the polygonal model of the function f, where the subscript h denotes the norm of the partition W. Thus we model S by the polygon S_h. For $k = 0, 1,..., L - 1$ define $S_{h,k}$ at the knots by requiring that

$$S_{h,k}(\xi_{k+1}) = I_1(\xi_{k+1}) + S_0 - \int_0^{\xi_{k+1}} (rI_0S_h)(x)\, dx \tag{2.2}$$

and at other points by

$$S_h(t) = S_{h,k}(t) = S_h(\xi_k) \frac{t - \xi_{k+1}}{\xi_k - \xi_{k+1}} + S_h(\xi_{k+1}) \frac{t - \xi_k}{\xi_{k-1} - \xi_k}.$$

In equation (2.2), we use $(rI_0S_h)(x)$ to denote the product $r(x)\, I_0(x)\, S_h(x)$. In general, given functions f, g and h we write $(fgh)(x) = f(x)\, g(x)\, h(x)$. We define $\tau_h(t) = \tau(t)$ and $I_h(t) = I_0(t)$, for $0 \leqslant t \leqslant t_0$.

On the interval $(t_0, c]$ we model the three functions τ, I and S by polygons which are extensions of the functions τ_h, I_h and S_h defined in the previous paragraph. Again, for notational simplicity the extensions are also denoted by τ_h, I_h and S_h, respectively.

For $k = L, L + 1,..., N - 1$ we have the representations

$$\tau_{h,k}(t) = d_{k,0} \frac{t - \xi_{k+1}}{\xi_k - \xi_{k+1}} + d_{k,1} \frac{t - \xi_k}{\xi_{k+1} - \xi_k}, \tag{2.3}$$

$$I_{h,k}(t) = e_{k,0} \frac{t - \xi_{k+1}}{\xi_k - \xi_{k+1}} + e_{k,1} \frac{t - \xi_k}{\xi_{k+1} - \xi_k}, \tag{2.4}$$

$$S_{h,k}(t) = f_{k,0} \frac{t - \xi_{k+1}}{\xi_k - \xi_{k+1}} + f_{k,1} \frac{t - \xi_k}{\xi_{k+1} - \xi_k}, \tag{2.5}$$

where $d_{k,0}$, $d_{k,1}$, $e_{k,0}$, $e_{k,1}$, $f_{k,0}$, $f_{k,1}$ are constants and t is in $[\xi_k, \xi_{k+1}]$. Since τ_h, I_h and S_h are continuous $d_{k,0} = \tau_{h,k-1}(\xi_k)$, $e_{k,0} = I_{h,k-1}(\xi_k)$, and $f_{k,0} = S_{h,k-1}(\xi_k)$. The unknown coefficients $d_{k,1}$, $e_{k,1}$, and $f_{k,1}$ in (2.3)–(2.5) are determined by requiring that τ_h, I_h and S_h satisfy (1.1)–(1.3) at the points ξ_{k+1}, $k = L, L+1, \ldots, N-1$, that is, by requiring

$$\int_{\tau_h(\xi_{k+1})}^{\xi_{k+1}} [\rho_1(x) + \rho_2(x)\, I_h(x)]\, dx = m, \tag{2.6}$$

$$I_h(\xi_{k+1}) = I_0(\xi_{k+1}) + \int_{\tau_h(\xi_{k+1}-\sigma)}^{\tau_h(\xi_{k+1})} (rI_hS_h)(x)\, dx, \tag{2.7}$$

$$S_h(\xi_{k+1}) = I_1(\xi_{k+1}) + S_0 - \int_{\tau_h(\xi_{k+1}-\sigma-\omega)}^{\xi_{k+1}} (rI_hS_h)(x)\, dx. \tag{2.8}$$

If the functions τ_h, I_h and S_h are known for $0 \leqslant t \leqslant \xi_{k-1}$, then (2.6)–(2.8) constitutes a system of three nonlinear equations for the three unknown coefficients $d_{k,1}$, $e_{k,1}$ and $f_{k,1}$. Once these coefficients are determined it follows from (2.3)–(2.5) that $\tau_{h,k}(\xi_{k+1}) = d_{k,1}$, $I_{h,k}(\xi_{k+1}) = e_{k,1}$, and $S_{h,k}(\xi_{k+1}) = f_{k,1}$. For given initial conditions, once S_h on the interval $[0, t_0]$ has been found, the coefficients $d_{k,1}$, $e_{k,1}$ and $f_{k,1}$, $k = L, L+1, \ldots, N-1$, are generated recursively by observing (2.6)–(2.8).

Notice that no special starting procedure is required by this method and that the knots W may be chosen arbitrarily except for ξ_L which must be t_0.

3. Theoretical Results

This section is devoted to theoretical results about the polygonal model introduced in Section 2. We will show that, under suitable hypotheses, there exists a polygon S_h which satisfies (2.2) on $[0, t_0]$ and that the system of equations (2.6)–(2.8) has a unique solution. We also prove that the sequences of functions $\{\tau_h\}$, $\{I_h\}$ and $\{S_h\}$ generated as described converge to τ, I and S, respectively, as h tends to zero. Before we proceed to these results we prove an interesting result concerning the Lipschitz continuity of τ, I and S.

Lemma 3.1. *If the given admissible function I_0 is Lipschitz continuous on the interval $[-\sigma, 0]$, then the functions τ, I and S are Lipschitz continuous on the finite interval $[0, c]$.*

Proof. It is obvious that I_0 as extended to the entire real line is Lipschitz continuous. It is also obvious from the definition that I_1 is Lipschitz continuous. Let λ_{I_0} denote a Lipschitz constant for I_0 and note that this constant is also a Lipschitz constant for I_1.

Given a continuous function f, defined on a finite interval $[0, b]$, we introduce the following notation:

$$f^*(b) = \max_{0 \leqslant t \leqslant b} f(t),$$

$$f_*(b) = \min_{0 \leqslant t \leqslant b} f(t).$$

Also, for notational convenience, let $Z(x)$ denote $\rho_1(x) + \rho_2(x) I(x)$. Throughout this proof we assume t_1 and t_2 are points in the interval $[0, c]$ with $t_1 < t_2$.

We now show that τ is Lipschitz continuous. There are three cases to be considered; namely, $t_2 \leqslant t_0$, $t_1 \leqslant t_0 < t_2$, and $t_0 \leqslant t_1$. The only nontrivial case occurs when $t_0 \leqslant t_1$. In this case the definition of τ implies

$$\int_{\tau(t_1)}^{\tau(t_2)} Z(x)\, dx = \int_{t_1}^{t_2} Z(x)\, dx.$$

Hoppensteadt and Waltman show in [6] that under our hypotheses $Z_*(c) > 0$. Thus

$$| \tau(t_2) - \tau(t_1)| \leqslant \frac{Z^*(c)}{Z_*(c)} \, | t_2 - t_1 | .$$

The Lipschitz continuity of I and S follows in an analogous manner.

Throughout this work we let λ_τ, λ_I and λ_S denote positive Lipschitz constants for τ, I and S, respectively.

We now show that there exists a polygon S_h on the interval $[0, t_0]$ which satisfies equation (2.2).

THEOREM 3.1. *Suppose $W = \{\xi_0, \xi_1, ..., \xi_L\}$ is a partition of $[0, t_0]$ with norm $h < 2/(I_0(0)\, r^*(t_0))$. Then there exists a unique positive polygon S_h in $Y(W)$ which is bounded above by P and satisfies equation* (1.2) *at the nodes, i.e.*

$$S_h(\xi_k) = I_1(\xi_k) + S_0 - \int_0^{\xi_k} (rI_h S_h)(x)\, dx, \qquad k = 1, 2, ..., L.$$

Proof. In this proof we use the facts that for $t \geqslant 0$, I_0 is nonincreasing and I_1 is nondecreasing. Also, recall that for $0 \leqslant t \leqslant t_0$, $I_h(t) = I_0(t)$.

The proof is by induction. For $j = 1$ we must have

$$S_h(\xi_1) + \int_0^{\xi_1} (rI_h S_h)(x)\, dx = I_1(\xi_1) + S_0 .$$

Replacing S_h by its representation on $[\xi_0, \xi_1]$ we have

$$f_{0,1} + \int_0^{\xi_1} r(x)\, I_h(x) \left[S_h(\xi_0) \frac{x - \xi_1}{\xi_0 - \xi_1} + f_{0,1} \frac{x - \xi_0}{\xi_1 - \xi_0} \right] dx = I_1(\xi_1) + S_0 .$$

This implies

$$f_{0,1} \left[1 + \int_0^{\xi_1} r(x)\, I_h(x) \frac{x - \xi_0}{\xi_1 - \xi_0}\, dx \right]$$
$$= I_1(\xi_1) + S_0 \left[1 - \int_0^{\xi_1} r(x)\, I_h(x) \frac{x - \xi_1}{\xi_0 - \xi_1}\, dx \right] .$$

The coefficient of $f_{0,1}$ is positive so this equation has a unique solution. Since $I_1 \geqslant 0$, $S_0 > 0$ and

$$\begin{aligned}\int_0^{\xi_1} r(x)\, I_h(x) \frac{x-\xi_1}{\xi_0-\xi_1}\, dx &\leqslant I_0(0)\, r^*(t_0) \frac{1}{\xi_0-\xi_1} \int_0^{\xi_1} (x-\xi_1)\, dx \\ &\leqslant \tfrac{1}{2} I_0(0)\, r^*(t_0)\, (\xi_1-\xi_0) \\ &< 1,\end{aligned}$$

the right-hand side is also positive. Thus the solution $f_{0,1}$ is positive.

The inductive step, which is made in the same manner, is omitted. Since $S_{h,k}(\xi_{k+1}) = f_{k,1} > 0$ for $k = 0, 1,\ldots, L-1$, the polygon, denoted S_h, joining the points $(\xi_k, S_h(\xi_k))$ is positive and has the desired properties. This concludes the proof of Theorem 3.1.

In the process of finding a bound for the quantity $|S(t) - S_h(t)|$ we will find the following elementary lemma, which is stated without proof, useful.

LEMMA 3.2. *Suppose f is a function defined on an interval $[a, b]$ and that $t_1 < t_2$ are points in $[a, b]$. Let L_f denote the linear interpolating polynomial through the points $(t_1, f(t_1))$ and $(t_2, f(t_2))$. Suppose the numbers $f_h(t_1)$ and $f_h(t_2)$ are given and let L_{f_h} denote the linear interpolating polynomial through the points $(t_1, f_h(t_1))$ and $(t_2, f_h(t_2))$. If f is Lipschitz continuous with Lipschitz constant λ_f, then, for $t_1 \leqslant t \leqslant t_2$,*

$$|f(t) - L_f(t)| \leqslant 2\lambda_f\, |t_2 - t_1|$$

and

$$|f(t) - L_{f_h}(t)| \leqslant 2\lambda_f\, |t_2 - t_1| + \max_{i=1,2}\{|f_h(t_i) - f(t_i)|\}.$$

In the proof of the next lemma and thereafter we use the following notation. Given a partition $W = \{\xi_0, \xi_1,\ldots, \xi_N\}$ of an interval $[0, a]$ and functions f and f_h defined on $[0, a]$, we define $E(f, f_h, j)$ by

$$E(f, f_h, j) = \max\{|f(\xi_i) - f_h(\xi_i)|,\ i = 0, 1, 2,\ldots, j\}.$$

In general f_h will be a polygonal model of f. When it is clear which function f_h is the polygonal model of f, the notation $E(f, f_h, j)$ is shortened to $E(f, j)$. We also define

$$\|f\|^{[a,b]} = \max_{t\in[a,b]} |f(t)|\,.$$

For the remainder of this section it is assumed that the given admissible function I_0 is Lipschitz continuous. We remind the reader that the Hoppensteadt and Waltman hypotheses are also assumed to hold and that the population size is constant.

LEMMA 3.3. *Suppose the number a satisfies $0 < a \leqslant t_0$ and $r^*(t_0)\, I_0(0)\, a < 1$, the partition*

$$0 = \xi_0 < \xi_1 < \cdots < \xi_{K-1} < a \leqslant \xi_K < \xi_{K+1} < \cdots < \xi_L = t_0$$

has norm $h < 1/(r^*(t_0)\, I_0(0))$, and the polygon S_h, constructed with knots ξ_0, $\xi_1, \ldots, \xi_L$, is positive, bounded above by P and satisfies (2.2). *Then for $0 \leqslant t \leqslant a$,*

$$| S(t) - S_h(t) | \leqslant Ch,$$

where C is a constant independent of h.

Proof. If $K = 1$, the proof is easy; thus assume $K > 1$. Since $I = I_h = I_0$ on the interval $[0, t_0]$, it follows from (1.2) and (2.2) that

$$S(\xi_k) - S_h(\xi_k) = \int_0^{\xi_k} (rI_0 S_h)(x)\, dx - \int_0^{\xi_k} (rI_0 S)(x)\, dx,$$

for $k = 1, 2, \ldots, K$. Taking absolute values, we obtain

$$\begin{aligned} &| S(\xi_k) - S_h(\xi_k) | \\ &\quad\leqslant r^*(t_0)\, I_0(0) \left| \int_0^{\xi_{k-1}} [S_h(x) - S(x)]\, dx + \int_{\xi_{k-1}}^{\xi_k} [S_h(x) - S(x)]\, dx \right| \\ &\quad\leqslant r^*(t_0)\, I_0(0) \{a \| S_h - S \|^{[0,\xi_{k-1}]} + P(\xi_k - \xi_{k-1})\}. \end{aligned}$$

The maximum value of $| S_h(t) - S(t) |$ occurs on some subinterval, say $[\xi_{j-1}, \xi_j]$. Applying Lemma 3.2 we find

$$\begin{aligned} \| S_h - S \|^{[0,\xi_{k-1}]} &\leqslant 2\lambda_S | \xi_j - \xi_{j-1} | + \max_{i=j-1,j} \{| S_h(\xi_i) - S(\xi_i) |\} \\ &\leqslant 2\lambda_S h + E(S, k-1). \end{aligned}$$

Using this we obtain

$$\begin{aligned} | S(\xi_k) - S_h(\xi_k) | &\leqslant r^*(t_0)\, I_0(0)\, [2a\lambda_S h + aE(S, k-1) + Ph] \\ &\leqslant Qh + ME(S, k-1), \end{aligned} \tag{3.1}$$

where $Q = r^*(t_0)\, I_0(0)\, [2a\lambda_S + P]$ and $M = r^*(t_0)\, I_0(0)\, a$. Since the right-hand side of inequality (3.1) is a nondecreasing function of k, we have

$$E(S, k) \leqslant Qh + ME(S, k-1), \qquad k = 1, 2, \ldots, K.$$

From this relation it is easily shown that

$$E(S, k) \leqslant \left[\sum_{j=0}^{k-1} M^j \right] Qh.$$

By hypothesis M is less than 1, so we conclude

$$E(S, k) \leqslant C_0 h, \qquad k = 1, 2, \ldots, K,$$

where $C_0 = Q \sum_{j=0}^{\infty} M^j = Q/(1 - M)$. It follows from Lemma 3.2 that

$$\| S - S_h \|^{[0,a]} \leqslant Ch$$

with $C = 2\lambda_S + C_0$ completing the proof of Lemma 3.3.

Note that the inequality in the conclusion of Lemma 3.3 holds on $[0, \xi_k]$. Next we extend the results of the last lemma to a larger interval.

LEMMA 3.4. *Suppose the numbers* a *and* b *satisfy* $0 < a < b \leqslant t_0$ *and* $2r^*(t_0) I_0(0)(b - a) < 1$; *the partition*

$$\begin{aligned} 0 = \xi_0 < \xi_1 < \cdots < \xi_{K-1} < a \leqslant \xi_K < \cdots < \xi_{J-1} < b \\ \leqslant \xi_J < \cdots < \xi_L = t_0 \end{aligned}$$

of $[0, t_0]$ *has norm* $h < b - a$; *the polygon* S_h, *constructed with knots* $\xi_0, \xi_1, \ldots, \xi_L$, *is positive, bounded above by* P *and satisfies* (2.2) *and* $\| S - S_h \|^{[0,a]} \leqslant C_0 h$ *for a constant* C_0. *Then there exists a constant* C_1 *such that*

$$\| S - S_h \|^{[0,b]} \leqslant C_1 h.$$

Proof. For $k = K, K + 1, \ldots, J$ we have

$$\begin{aligned} S(\xi_k) - S_h(\xi_k) = & \int_0^{\xi_K} r(x) I_0(x) [S_h(x) - S(x)] \, dx \\ & + \int_{\xi_K}^{\xi_{k-1}} r(x) I_0(x) [S_h(x) - S(x)] \, dx \\ & + \int_{\xi_{k-1}}^{\xi_k} r(x) I_0(x) [S_h(x) - S(x)] \, dx. \end{aligned}$$

Taking absolute values we obtain

$$\begin{aligned} | S(\xi_k) - S_h(\xi_k) | \leqslant & \; r^*(t_0) I_0(0) t_0 C_0 h \\ & + r^*(t_0) I_0(0) \| S_h - S \|^{[0,\xi_{k-1}]} (\xi_{k-1} - \xi_K) \\ & + r^*(t_0) I_0(0) Ph \\ \leqslant & \; r^*(t_0) I_0(0) t_0 C_0 h + r^*(t_0) I_0(0) Ph \\ & + r^*(t_0) I_0(0) [2\lambda_S h + E(S, k - 1)] \, 2(b - a) \\ \leqslant & \; Qh + ME(S, k - 1), \end{aligned}$$

where $Q = r^*(t_0) I_0(0) [t_0 C_0 + 4\lambda_S(b-a) P]$ and $M = 2r^*(t_0) I_0(0)(b-a)$. As in the proof of Lemma 3.3, it follows that

$$E(S, k) \leqslant Qh + ME(S, k-1).$$

From this relation it is easily shown that for $j = 0, 1,..., J - K$,

$$E(S, K+j) \leqslant \frac{Q}{1-M} h + MC_0 h.$$

Applying Lemma 3.2 we obtain

$$\| S - S_h \|^{[a,b]} \leqslant 2\lambda_S h + E(S, J)$$

$$\leqslant \left(2\lambda_S + \frac{Q}{1-M} + MC_0\right) h.$$

Taking $C_1 = \max\{C_0, 2\lambda_S + Q/(1-M) + MC_0\}$, we have $\| S - S_h \|^{[0,b]} \leqslant C_1 h$, which completes the proof.

By applying 3.4 repeatedly, we extend this result to the interval $[0, t_0]$.

THEOREM 3.2. *Suppose the partition* $W = \{\xi_0, \xi_1,..., \xi_L\}$ *of the interval* $[0, t_0]$ *has the norm* $h < t_0/N_0$, *where* N_0 *is a positive integer such that* $t_0/N_0 < 1/(2r^*(t_0) I_0(0))$. *In addition, suppose the polygon* S_h, *constructed with knots* $\xi_0, \xi_1,..., \xi_L$ *is positive, bounded above by P and satisfies* (2.2). *Then there exists a constant C such that*

$$\| S - S_h \|^{[0,t_0]} \leqslant Ch.$$

Proof. Let $d = t_0/N_0$. By Lemma 3.3, there is a constant C_1 such that $\| S - S_h \|^{[0,d]} \leqslant C_1 h$. Applying Lemma 3.4 with $a = d$ and $b = 2d$ we find a constant C_2 such that $\| S - S_h \|^{[0,2d]} \leqslant C_2 h$. By applying Lemma 3.4 $N_0 - 2$ more times we obtain the conclusion of the theorem.

On the interval $[0, t_0]$, τ_h and I_h are known functions. For a larger interval, say $[0, c]$ where $c > t_0$, we model τ and I by polygons denoted τ_h and I_h, respectively. We now state and prove a theorem which establishes the existence and uniqueness of polygons τ_h, I_h and S_h satisfying (2.6)–(2.8).

THEOREM 3.3. *Suppose* $c > 0$ *and* $\epsilon > 0$ *are given and the partition* $W = \{\xi_0, \xi_1,..., \xi_L\}$ *of the interval* $[0, t_0]$ *has norm* $h < 2/(I_0(0) r^*(t_0))$. *Let* S_h *denote the polygonal function of Theorem* 3.1. *Then there exists a partition*

$$t_0 = \xi_L < \xi_{L+1} < \cdots < \xi_N = c$$

of $[t_0, c]$ *with* $\xi_{k+1} - \xi_k < \epsilon$, $k = L, L+1,..., N$, *and corresponding unique* *positive polygonal functions τ_h, I_h and S_h, which are extensions of τ, I_0 and S_h above, respectively, with knots $\xi_L, \xi_{L+1}, \ldots, \xi_N$, which satisfy*

$$\int_{\tau_h(\xi_k)}^{\xi_k} [\rho_1(x) + \rho_2(x) I_h(x)]\, dx = m, \tag{3.2}$$

$$S_h(\xi_k) = I_1(\xi_k) + S_0 - \int_{\tau_h(\xi_k-\sigma-\omega)}^{\xi_k} (rI_hS_h)(x)\, dx, \tag{3.3}$$

$$I_h(\xi_k) = I_0(\xi_k) + \int_{\tau_h(\xi_k-\sigma)}^{\tau(\xi_k)} (rI_hS_h)(x)\, dx, \tag{3.4}$$

$k = L+1, L+2, \ldots, N$. Furthermore, I_h and S_h are bounded above by P.

Proof. This proof is patterned after the existence proof given by Hoppensteadt and Waltman in [6].

Recall that on $[0, t_0]$, $\tau_h(t) = 0$ and $I_h(t) = I_0(t)$. Suppose that condition H1 holds with $I_0(t_0) > 0$ and that τ_h, I_h and S_h have been found on $[0, \xi_k]$, $\xi_k \geqslant t_0$, with $0 < I_h \leqslant P$ and $0 < S_h \leqslant P$. We wish to show that τ_h, I_h and S_h can be continued to a larger interval $[0, \xi_{k+1}]$ in a unique way.

Choose a point $\xi_{k+1} > \xi_k$ such that

$$\xi_{k+1} - \xi_k < \min\left\{\sigma, \frac{1}{Pr^*(c)}, \frac{m}{\rho_1^*(c) + \rho_2^*(c)\, P}, \epsilon, \frac{\rho_{2*}(c)\, I_{h*}(\xi_k)}{r^*(c)\, P^2 \rho_2^*(c)}\right\}. \tag{3.5}$$

If $\xi_k = t_0$, constrain ξ_{k+1} further by requiring that ξ_{k+1} be such that $2I_0(\xi_{k+1}) > I_0(t_0)$. Denote the set of polygonal functions with knots only at ξ_k and ξ_{k+1} by Y. For ϕ in Y define the norm of ϕ by

$$\|\phi\| = \max\{|\phi(t)|, \xi_k \leqslant t \leqslant \xi_{k+1}\}.$$

Let

$$M = \{\phi \mid \phi \in Y, \gamma \leqslant \phi(\xi_{k+1}) \leqslant P, \phi(\xi_k) = I_h(\xi_k)\},$$

where

$$\begin{aligned} \gamma &= \tfrac{1}{2} I_0(t_0), && \xi_k = t_0 \\ &= r_*(\xi_k)\, S_{h*}(\xi_k)\, I_{h*}(\xi_k)\, [\tau_h(\xi_k) - \tau_h(\xi_{k+1} - \sigma)], && \xi_k > t_0 . \end{aligned}$$

For $t_0 < \xi_j \leqslant \xi_k$, $\tau_h(\xi_j)$ is defined by

$$\int_{\tau_h(\xi_j)}^{\xi_j} [\rho_1(x) + \rho_2(x) I_h(x)]\, dx = m.$$

The integrand is positive, therefore $\tau_h(t)$ is increasing for $t_0 \leqslant t \leqslant \xi_k$.

By the choice of ξ_{k+1} we have $\xi_{k+1} - \sigma < \xi_k$, which implies $\tau_h(\xi_{k+1} - \sigma) < \tau_h(\xi_k)$. From this it follows that γ is positive for $\xi_k > t_0$. If $\xi_k = t_0$, γ is positive by hypothesis.

Equip M with the metric $\rho(\phi_1, \phi_2) = \| \phi_1 - \phi_2 \|$. Under this metric M is a complete metric space.

Define a mapping U from M into Y by requiring that $U\phi(\xi_{k+1})$ satisfy

$$\int_{U\phi(\xi_{k+1})}^{\xi_k} [\rho_1(x) + \rho_2(x) I_h(x)]\, dx + \int_{\xi_k}^{\xi_{k+1}} [\rho_1(x) + \rho_2(x)\, \phi(x)]\, dx = m \qquad (3.6)$$

and $U\phi(\xi_k) = \tau_h(\xi_k)$. Since $\rho_2(x)\, \phi(x) > 0$ and $\xi_{k+1} - \xi_k < m/(\rho_1^*(c) + \rho_2^*(c)\, P)$, these requirements uniquely define $U\phi$.

The important properties of $U\phi$ are established in the following lemma which is stated without proof.

LEMMA 3.5. *U is a continuous mapping of M into Y with $\tau_h(\xi_k) \leqslant (U\phi)(t) \leqslant \xi_k$ for $\xi_k \leqslant t \leqslant \xi_{k+1}$.*

Next define a mapping T on M into Y by requiring that $T\phi(\xi_k) = I_h(\xi_k)$ and

$$T\phi(\xi_{k+1}) = I_0(\xi_{k+1}) + \int_{\tau_h(\xi_{k+1} - \sigma)}^{U\phi(\xi_{k+1})} (rI_h S_h)(x)\, dx. \qquad (3.7)$$

The essential property of T is established in the next lemma which is also stated without proof.

LEMMA 3.6. *The mapping T is a contraction mapping of M into itself.*

Since T is a contraction mapping on M, Banach's fixed point theorem implies the existence of a unique point ϕ such that $T\phi = \phi$. Extend the functions τ_h and I_h, known on $[0, \xi_k]$, to $[0, \xi_{k+1}]$ by defining them on $[\xi_k, \xi_{k+1}]$ as $I_h(t) = \phi(t)$ and $\tau_h(t) = U\phi(t)$. From equations (3.6) and (3.7) we see that this extension satisfies equations (3.2) and (3.4).

With $\tau_h(\xi_{k+1})$ and $I_h(\xi_{k+1})$ known, we show the existence and uniqueness of $S_h(\xi_{k+1})$ in the same manner as in the proof of Theorem 3.1. Thus τ_h, I_h and S_h have been continued to the point ξ_{k+1}.

We now wish to show that the above extension process can be used to extend τ_h, I_h and S_h to the interval $[0, c]$ in a finite number of steps.

Let Ω denote the set of all $x \leqslant c$ such that τ_h, I_h and S_h cannot be continued, in a finite number of steps and in a unique way, to $t = x$. If Ω is empty then this proof is complete. If not, let $\bar{t} = \inf \Omega$ and choose a point $\xi_j < \bar{t}$ such that

$$\bar{t} - \xi_j < \min \left\{ \sigma, \frac{m}{\rho_1^*(c) + \rho_2^*(c)\, P} \right\}.$$

By definition of $\bar{t}$, the functions τ_h, I_h and S_h can be continued to a point $\xi_i > \xi_j$ such that

$$\bar{t} - \xi_i < \min \left\{ \frac{\rho_{2*}(c)\, I_{h*}(\xi_j)}{2r^*(c)\, P^2 \rho_2^*(c)}, \frac{1}{Pr^*(c)}, \epsilon \right\}.$$

To show that τ_h, I_h and S_h can be continued to $\bar{t}$, we repeat the extension argument given above with $\xi_k = \xi_i$ and $\xi_{k+1} = \bar{t}$. If $\bar{t} = c$ then the proof is complete. If not, using the same argument again, we extend τ_h, I_h and S_h past $\bar{t}$. This contradicts the definition of $\bar{t}$, thus τ_h, I_h and S_h may be extended to c in a finite number of steps.

This completes the proof with H1 holding and $I_0(t_0) > 0$. The proof with H2 holding is similar. We omit that part of the proof.

In the preceding proof, the functions τ_h, I_h and S_h were assumed to exist on $[0, \xi_k]$ and it was shown that these functions could be continued to $[0, \xi_{k+1}]$ where $\xi_{k+1} > \xi_k$. Upon consideration of the proof we note that ξ_{k+1} is chosen from a continuum of possible values. That is, if the number x could be chosen to be ξ_{k+1}, then any number in $(\xi_k, x]$ could be chosen as ξ_{k+1}. Thus any point may be specified in advance to be a knot of the polygons τ_h, I_h and S_h.

Given a finite interval $[0, c]$, we are interested in generating sequences of polygons $\{\tau_h\}$, $\{I_h\}$ and $\{S_h\}$ defined on $[0, c]$ and in showing that these sequences converge to τ, I and S, respectively. We use the following scheme to generate sequences of polygons which have the desired convergence property. Let $\{\epsilon_n\}$ denote a sequence of positive numbers such that ϵ_n tends to zero as n tends to infinity. By the preceding theorem, for each n, we can construct a partition W_n of $[0, c]$, with norm less than ϵ_n, and corresponding unique polygons $\tau_{h,n}$, $I_{h,n}$ and $S_{h,n}$. In fact, it follows from the preceding paragraph that for each n we can construct many partitions and corresponding polygons. For our purposes, we choose any one of these and let it correspond to ϵ_n. In this way we obtain a sequence of partitions $\{W_n\}$ and corresponding sequences of polygons $\{\tau_{h,n}\}$, $\{I_{h,n}\}$ and $\{S_{h,n}\}$. Notice that the norms of the partitions approach zero as n tends to infinity. We wish to show that as $n \to \infty$, $\tau_{h,n} \to \tau$, $I_{h,n} \to I$ and $S_{h,n} \to S$. The method of attack is the same as that which led to Theorem 3.2.

LEMMA 3.7. *Suppose c and M are constants with $c > t_0$ and $4M < 1$. Let*

$$H = \max \left\{ \frac{\rho_2^*(c)}{Z_*(c)}, r^*(c)\, P^2, r^*(c)\, P \left(1 + \frac{P\rho_2^*(c)}{Z_*(c)}\right) \right\},$$

let a be a constant such that $t_0 \leqslant a < c$, and let $a < b \leqslant \min\{a + M/H, c\}$. Suppose the polygonal functions τ_h, I_h and S_h have been constructed, as described in Section 2, with knots

$$0 = \xi_0 < \xi_1 < \cdots < \xi_{K-1} < a \leqslant \xi_K < \cdots < \xi_N = c,$$

where the partition $\{\xi_0, \xi_1, \ldots, \xi_N\}$ has norm $h < \min\{\sigma, M/2H, b - a\}$. Finally, suppose that I_h and S_h are nonnegative and bounded above by P, that $\tau_h(\xi_j) \leqslant \xi_j$, $j = K, K+1, \ldots, N$, and that $\| \tau - \tau_h \|^{[0,a]} \leqslant C_0 h$, $\| I - I_h \|^{[0,a]} \leqslant C_0 h$ and $\| S - S_h \|^{[0,a]} \leqslant C_0 h$, where C_0 is a constant. Then there exists a constant C_1 such that $\| \tau - \tau_h \|^{[0,b]} \leqslant C_1 h$, $\| I - I_h \|^{[0,b]} \leqslant C_1 h$ and $\| S - S_h \|^{[0,b]} \leqslant C_1 h$.

Proof. Let J be an integer such that $\xi_{J-1} < b \leqslant \xi_J$. The restrictions on h imply that $\xi_{J-1} \geqslant a$ and $\xi_{J-1} - \xi_{K-1} < 2M/H$. For $j = K, k+1,..., J$, we have

$$\int_{\tau(\xi_j)}^{\xi_j} Z(x)\,dx = \int_{\tau_h(\xi_j)}^{\xi_j} [\rho_1(x) + \rho_2(x)\,I_h(x)]\,dx,$$

which is equivalent to

$$\int_{\tau(\xi_j)}^{\tau_h(\xi_j)} Z(x)\,dx = -\int_{\tau_h(\xi_j)}^{\xi_j} \rho_2(x)\,[I(x) - I_h(x)]\,dx.$$

Applying the mean-value theorem to the first integral, we find a point x_1 between $\tau(\xi_j)$ and $\tau_h(\xi_j)$ such that

$$\tau(\xi_j) - \tau_h(\xi_j) = \frac{1}{Z(x_1)} \int_{\tau_h(\xi_j)}^{\xi_j} \rho_2(x)\,[I(x) - I_h(x)]\,dx. \tag{3.9}$$

We consider two cases. First we suppose that $\tau_h(\xi_j) \leqslant \xi_K$ and take absolute values in equation (3.9) to find

$$\begin{aligned} &|\tau(\xi_j) - \tau_h(\xi_j)| \\ &\quad\leqslant \frac{1}{Z_*(c)} \left| \int_{\tau_h(\xi_j)}^{\xi_K} \rho_2(x)\,[I(x) - I_h(x)]\,dx + \int_{\xi_K}^{\xi_{j-1}} \rho_2(x)\,[I(x) - I_h(x)]\,dx \right. \\ &\qquad \left. + \int_{\xi_{j-1}}^{\xi_j} \rho_2(x)\,[I(x) - I_h(x)]\,dx \right| \\ &\quad\leqslant \frac{1}{Z_*(c)} [\rho_2^*(c)\,cC_0 h + \rho_2^*(c)\,(\xi_{j-1} - \xi_K)\,\| I - I_h \|^{[0,\xi_{j-1}]} + \rho_2^*(c)\,Ph]. \end{aligned}$$

By Lemma 3.2

$$\| I - I_h \|^{[0,\xi_{j-1}]} \leqslant 2\lambda_I h + E(I, j-1),$$

therefore

$$|\tau(\xi_j) - \tau_h(\xi_j)| \leqslant \bar{Q}_\tau h + \frac{2M\rho_2^*(c)}{HZ_*(c)}\,E(I, j-1), \tag{3.10}$$

where

$$\bar{Q}_\tau = \frac{\rho_2^*(c)}{Z_*(c)} \left[cC_0 + \frac{4M}{H}\lambda_I + P\right].$$

Let $Q_\tau = \max\{\bar{Q}_\tau, C_0\}$, then we have

$$E(\tau, j) \leqslant Q_\tau h + \frac{2M\rho_2^*(c)}{HZ_*(c)}\,E(I, j-1). \tag{3.11}$$

For the second case we assume $\tau_h(\xi_j) > \xi_K$ and we find

$$| \tau(\xi_j) - \tau_h(\xi_j)| \leqslant \frac{1}{Z_*(c)} \left[\int_{\xi_K}^{\xi_{j-1}} \rho_2(x) \mid I(x) - I_h(x)| \, dx + \int_{\xi_{j-1}}^{\xi_j} \rho_2(x) \mid I(x) - I_h(x)| \, dx \right].$$

Thus we see that in this case also equation (3.10) holds.

Next we consider I. Using (1.3) and (2.7) we find, for $j = K + 1, K + 2, \ldots, J$,

$$\begin{aligned} I(\xi_j) - I_h(\xi_j) &= \int_{\tau(\xi_j - \sigma)}^{\tau(\xi_j)} (rIS)(x) \, dx - \int_{\tau_h(\xi_j - \sigma)}^{\tau_h(\xi_j)} (rI_hS_h)(x) \, dx \\ &= G_1 + G_2 + G_3 , \end{aligned}$$

where

$$G_1 = \int_{\tau(\xi_j - \sigma)}^{\tau_h(\xi_j - \sigma)} (rIS)(x) \, dx,$$

$$G_2 = \int_{\tau_h(\xi_j)}^{\tau(\xi_j)} (rIS)(x) \, dx,$$

$$G_3 = \int_{\tau_h(\xi_j - \sigma)}^{\tau_h(\xi_j)} r(x) \, [I(x) \, S(x) - I_h(x) \, S_h(x)] \, dx.$$

Using Lemma 3.2 we see that

$$| G_1 | \leqslant \tfrac{1}{4} r^*(c) \, P^2 [2\lambda_\tau h + E(\tau, j - 1)].$$

Now we use equation (3.11) to obtain

$$| G_1 | \leqslant \frac{1}{2} r^*(c) \, P^2 \lambda_\tau h + \frac{1}{4} r^*(c) \, P^2 \left[Q_\tau h + \frac{2M\rho_2^*(c)}{HZ_*(c)} E(I, j - 1) \right]. \tag{3.12}$$

For the second integral we have

$$\begin{aligned} | G_2 | &\leqslant \frac{1}{4} r^*(c) \, P^2 \mid \tau(\xi_j) - \tau_h(\xi_j)| \\ &\leqslant \frac{1}{4} r^*(c) \, P^2 \left[Q_\tau h + \frac{2M\rho_2^*(c)}{HZ_*(c)} E(I, j - 1) \right]. \end{aligned} \tag{3.13}$$

To bound G_3, we will use

$$\begin{aligned} \| IS - I_h S_h \|^{[0,a]} &\leqslant \| S \|^{[0,a]} \| I - I_h \|^{[0,a]} \\ &\quad + \| I_h \|^{[0,a]} \| S - S_h \|^{[0,a]} \\ &\leqslant 2PC_0 h, \end{aligned}$$

and

$$\| IS - I_h S_h \|^{[0,\xi_j-1]} \leqslant P[2\lambda_I h + E(I, j-1) + 2\lambda_S h + E(S, j-1)].$$

If $\tau_h(\xi_j) > \xi_K$ and $\tau_h(\xi_j - \sigma) \leqslant \xi_K$, then

$$\begin{aligned}
| G_3 | &\leqslant \left| \int_{\tau_h(\xi_j-\sigma)}^{\xi_K} r(x) [I(x) S(x) - I_h(x) S_h(x)] \, dx \right| \\
&\quad + \left| \int_{\xi_K}^{\tau_h(\xi_j)} r(x) [I(x) S(x) - I_h(x) S_h(x)] \, dx \right| \\
&\leqslant | \xi_K - \tau_h(\xi_j - \sigma)| \, r^*(c) \, 2PC_0 h \\
&\quad + \int_{\xi_K}^{\xi_{j-1}} r(x) \, | I(x) S(x) - I_h(x) S_h(x)| \, dx \\
&\quad + \int_{\xi_{j-1}}^{\xi_j} r(x) \, | I(x) S(x) - I_h(x) S_h(x)| \, dx \\
&\leqslant 2cC_0 P r^*(c) \, h + r^*(c) \, P^2 h \\
&\quad + (\xi_{j-1} - \xi_K) \, r^*(c) \, P[2\lambda_I h + E(I, j-1) + 2\lambda_S h + E(S, j-1)] \\
&\leqslant r^*(c) \, P \left[2cC_0 + \frac{4M}{H} (\lambda_I + \lambda_S) + P \right] h \\
&\quad + \frac{2M}{H} r^*(c) \, P[E(I, j-1) + E(S, j-1)]. \qquad (3.14)
\end{aligned}$$

If $\tau_h(\xi_j) > \xi_K$ and $\tau_h(\xi_j - \sigma) > \xi$, then

$$| G_3 | \leqslant \int_{\xi_K}^{\tau_h(\xi_j)} r(x) \, | I(x) S(x) - I_h(x) S_h(x)| \, dx.$$

From the above we see that equation (3.14) also holds for this case. If $\tau_h(\xi_j) \leqslant \xi_K$ then

$$| G_3 | \leqslant 2cr^*(c) \, PC_0 h.$$

Thus equation (3.14) holds in all cases. Using equations (3.12)–(3.14) we obtain

$$\begin{aligned}
&| I(\xi_j) - I_h(\xi_j)| \\
&\quad \leqslant \bar{Q}_I h + r^*(c) \, P \frac{2M}{H} E(S, j-1) + r^*(c) \, P \frac{2M}{H} \left[1 + \frac{P\rho_2^*(c)}{2Z_*(c)} \right] E(I, j-1), \qquad (3.15)
\end{aligned}$$

where

$$\bar{Q}_I = r^*(c) \, P \left[\frac{1}{2} P(\lambda_\tau + Q_\tau) + 2cC_0 + \frac{4M}{H} (\lambda_I + \lambda_S) + P \right].$$

Next, let us consider S. Using equations (1.2) and (2.8) we see that

$$S(\xi_j) - S_h(\xi_j) = \int_{\tau_h(\xi_j-\sigma-\omega)}^{\xi_j} (rI_hS_h)(x)\,dx - \int_{\tau_h(\xi_j-\sigma-\omega)}^{\xi_j} (rIS)(x)\,dx$$

$$= G_4 + G_5 + G_6\,,$$

where

$$G_4 = \int_{\tau_h(\xi_j-\sigma-\omega)}^{\tau(\xi_j-\sigma-\omega)} (rIS)(x)\,dx,$$

$$G_5 = \int_{\tau_h(\xi_j-\sigma-\omega)}^{\xi_{j-1}} r(x)\,[I_h(x)\,S_h(x) - I(x)\,S(x)]\,dx,$$

$$G_6 = \int_{\xi_{j-1}}^{\xi_j} r(x)\,[I_h(x)\,S_h(x) - I(x)\,S(x)]\,dx.$$

Using the same procedure as that used to obtain equation (3.12), we obtain

$$|\,G_4\,| \leqslant \frac{1}{2}\,r^*(c)\,P^2\lambda_\tau h + \frac{1}{4}\,r^*(c)\,P^2\left[Q_\tau h + \frac{2M\rho_2^*(c)}{HZ_*(c)}\,E(I, j-1)\right].$$

In order to bound G_5, we consider two cases. If $\tau_h(\xi_j - \sigma - \omega) \leqslant \xi_K$ then

$$|\,G_5\,| \leqslant \left|\int_{\tau_h(\xi_j-\sigma-\omega)}^{\xi_K} r(x)\,[I_h(x)\,S_h(x) - I(x)\,S(x)]\,dx\right|$$

$$+ \left|\int_{\xi_K}^{\xi_{j-1}} r(x)\,[I_h(x)\,S_h(x) - I(x)\,S(x)]\,dx\right|$$

$$\leqslant 2r^*(c)\,PC_0h\,|\,\xi_K - \tau_h(\xi_j - \sigma - \omega)|$$

$$+ r^*(c)\,\frac{2M}{H}\,P[2\lambda_S h + E(S, j-1) + 2\lambda_I h + E(I, j-1)]$$

$$\leqslant 2r^*(c)\,P\left[cC_0 + \frac{2M}{H}\,(\lambda_S + \lambda_I)\right]h$$

$$+ r^*(c)\,\frac{2M}{H}\,P[E(S, j-1) + E(I, j-1)].$$

If $\tau_h(\xi_j - \sigma - \omega) > \xi_K$ then

$$|\,G_5\,| \leqslant \int_{\xi_K}^{\xi_{j-1}} r(x)\,|\,I_h(x)\,S_h(x) - I(x)\,S(x)|\,dx.$$

Thus the bound obtained in the first case holds for this case as well. Finally

$$| G_6 | \leqslant r^*(c)\, P^2(\xi_j - \xi_{j-1}).$$

Combining the bounds for G_4, G_5 and G_6 we find

$$\begin{aligned} &| S(\xi_j) - S_h(\xi_j)| \\ &\quad \leqslant \bar{Q}_S h + r^*(c)\, P \frac{2M}{H} E(S, j-1) + r^*(c)\, P \frac{2M}{H} \left[1 + \frac{P\rho_2^*(c)}{4Z_*(c)}\right] E(I, j-1), \end{aligned} \tag{3.16}$$

where

$$\bar{Q}_S = r^*(c)\, P \left[\frac{1}{2} P\lambda_\tau + \frac{1}{4} PQ_\tau + 2cC_0 + \frac{4M}{H} (\lambda_S + \lambda_I) + P\right].$$

By the choice of b, we have from (3.11), (3.15) and (3.16) that

$$E(\tau, j) \leqslant Q_\tau h + \alpha E(I, j-1), \tag{3.17}$$

$$E(I, j) \leqslant Q_I h + \alpha E(S, j-1) + \alpha E(I, j-I), \tag{3.18}$$

$$E(S, j) \leqslant Q_S h + \alpha E(S, j-1) + \alpha E(I, j-I), \tag{3.19}$$

where $\alpha = 2M < \frac{1}{2}$, $Q_I = \max\{\bar{Q}_I, C_0\}$ and $Q_S = \max\{\bar{Q}_S, C_0\}$. Using equations (3.17)–(3.19), it is easily shown, by finite induction, that

$$E(\tau, j) \leqslant Ah, \qquad E(I, j) \leqslant Bh \qquad \text{and} \qquad E(S, j) \leqslant Dh,$$

where

$$A = Q_\tau + \alpha Q_I + \sum_{k=2}^{\infty} 2^{k-2} \alpha^k (Q_I + Q_S) + C_0,$$

$$B = Q_I + \sum_{k=1}^{\infty} 2^{k-1} \alpha^k (Q_I + Q_S) + C_0,$$

$$D = Q_S + \sum_{k=1}^{\infty} 2^{k-1} \alpha^k (Q_I + Q_S) + C_0.$$

Since τ, I and S are Lipschitz continuous, the inequalities

$$\begin{aligned} | \tau(t) - \tau_h(t)| &\leqslant 2\lambda_\tau h + E(\tau, J) \\ &\leqslant (2\lambda_\tau + A)\, h, \\ | I(t) - I_h(t)| &\leqslant (2\lambda_I + B)\, h, \\ | S(t) - S_h(t)| &\leqslant (2\lambda_S + D)\, h, \end{aligned}$$

hold for $a < t \leqslant b$. We complete the proof of Lemma 3.6 by letting

$$C_1 = \max\{2\lambda_\tau + A, 2\lambda_I + B, 2\lambda_S + D\}.$$

THEOREM 3.4. *Suppose c is a positive number greater than t_0. Let $\{\epsilon_n\}$ denote a sequence of positive numbers which conveges to zero. Associate with each ϵ_n a partition W_n of $[0, c]$, given by*

$$0 = \xi_{n,1} < \xi_{n,2} < \cdots < \xi_{n,L_n} = t_0 < \cdots < \xi_{n,N_n} = c,$$

with norm less than ϵ_n, and corresponding unique nonnegative functions $\tau_{h,n}$, $I_{h,n}$ and $S_{h,n}$ which satisfy the following

(i) *For $0 \leqslant t \leqslant t_0$, $\tau_{h,n}(t) = 0$, $I_{h,n}(t) = I_0(t)$ and $S_{h,n}$ is a positive polygonal function with knots $\xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,L_n}$ which satisfies* (2.2).

(ii) *For $t_0 < t \leqslant c$, $\tau_{h,n}$, $I_{h,n}$ and $S_{h,n}$ are nonnegative polygonal functions, with knots $\xi_{n,L_n+1}, \xi_{n,L_n+2}, \ldots, \xi_{n,N_n}$, which satisfy* (2.6)–(2.8).

Then the sequences of functions $\{\tau_{h,n}\}$, $\{I_{h,n}\}$ and $\{S_{h,n}\}$ converge uniformly to τ, I and S, respectively.

Proof. For N_0 sufficiently large and $n \geqslant N_0$, the hypothesis of Theorem (3.2) is satisfied. Thus for $n \geqslant N_0$, there exists a constant C_0 such that $\| S - S_{h,n} \|^{[0,t_0]} \leqslant C_0\epsilon_n$. For $t > t_0$, let k denote a positive integer such that $d = (c - t_0)/k < M/H$. Then for each n sufficiently large the hypothesis of Lemma 3.7 is satisfied with $a = t_0$ and $b = t_0 + d$. Thus there exists a constant C_1 such that $\| \tau - \tau_{h,n} \|^{[0,t_0+d]} \leqslant C_1\epsilon_n$, $\| I - I_{h,n} \|^{[0,t_0+d]} \leqslant C_1\epsilon_n$ and $\| S - S_{h,n} \|^{[0,t_0+d]} \leqslant C_1\epsilon_n$. Applying Lemma 3.7 $k - 1$ more times we find a constant C such that $\| \tau - \tau_{h,n} \|^{[0,c]} \leqslant C\epsilon_n$, $\| I - I_{h,n} \|^{[0,c]} \leqslant C\epsilon_n$ and $\| S - S_{h,n} \|^{[0,c]} \leqslant C\epsilon_n$. Since C is independent of n we have the conclusion of Theorem 3.4.

4. CONCLUSION

The method described in Section 2 has been implemented on a digital computer by replacing the integrals involved by quadrature rules which are generalizations of the product-type quadrature rules developed by Boland and Duris [1]. These quadrature rules are well suited to this problem since they are designed to approximate the integral of the product of two or more functions. In the implementation a product-type rule was used which integrates exactly the product of three polygons. Details concerning this implementation as well as examples which include values of τ, I and S appear in [2].

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